The Science, Art, and Teaching of Mathematics

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The creation is the artwork of God, displaying his glory. In mathematical science we reflect the thoughts of God, applying our mathematical art to God's creation. Children best learn mathematics in ways that emphasize experience.

The Science of Mathematics is the study of the mathematical order of the creation. As such, it is based ultimately in the mathematical nature of the creative and sustaining Logos.[1] The Art of Mathematics is the development of Mathematical Theory according to some more or less developed axiomatic system. The axioms are designed to reflect the thoughts of God. As art, the theory may be developed beyond what is applicable by man to the creation in order to reflect the brilliance of God.

This lack of applicability of the art may be due to temporal lack of knowledge. Further, however, God need not have created exhaustively to display all aspects of His mathematical nature. Therefore, there may exist in the mind of God logically consistent axiomatic systems not displayed anywhere in the creation.

We say “axiomatic system” not implying that the mind of God is bound by axioms, but that axioms in themselves are the thoughts of God. They are “consistent” because the eternal Logos thinks them. Our axioms, in as much as they are axioms at all, are representations of the thoughts of God. Therefore we speak of them as “art”. Here we use the term “art” to mean that which humans create to display form and beauty. Our use of the term “art” is unusual in that the medium of our constructions is our thoughts. But we use the term “art” to emphasize the reconstructive role of man as the subject of knowledge. Though Professor Van Til usually used the term “analogical thought” to describe the thoughts of man as subject, our emphasis on man's reconstructive thought is

1 For a discussion of non-Christian views of mathematics, see Professor Vern S. Poythress, “A Biblical View of Mathematics”, in *Foundations of Christian Scholarship*, ed. Gary North (Ross House, 1979), pp. 158-188. In addition, Dr. Poythress provides in the article an excellent discussion of a Christian view of mathematics for which this author is much indebted.

found in his work. Consider for example, the following quote from Professor Van Til's *Defense of the Faith*.

> In the first place there is the Adamic Consciousness. When man was first created he was perfect. He recognized the fact that he was a creature; he was actually normal. He wanted to be nothing but a re-interpreter of the interpretation of God. He was receptive to God's revelation which appeared within him and round about him; he would reconstruct this revelation. He was receptively reconstructive. For that reason he had real though not comprehensive unity in his experience.[2]

We do not even know comprehensively the image of the Logos which we bear. Therefore, our axioms are representations of the logos of our thoughts which reflect the thoughts of God. For example, the Zermelo-Fraenkel axiom that given a set there exists the set of all subsets of that set is a representation of the thoughts of the Logos as far as it is an axiom at all. After all, does not the Logos know all the parts of what it has made? Another example is the Axiom of Choice. This axiom states that if there exists an infinite collection of nonempty sets, then there exists a set containing one element from each of the members of the infinite collection. Some mathematicians reject the Axiom of Choice, since human beings may be able to build such a set from finite collections, but we cannot know infinite collections in this manner. But it is the belief of this author that the infinite Logos knows all things and the collections of all things. Hence, this axiom is acceptable.

In addition, as there most certainly exist in the mind of God axiomatic systems unknowable to man, some of these may be displayed in that part of the creation over which man has not been granted dominion in the Lord. For example, we have no basis on which to build an advanced study of the order of angelic beings, or even to presume that such a physics involves a metric space. We simply cannot speak scientifically of such things with mathematics. Though the Word of God written records such beings, and the wings of the cherubim can be counted, it has not been given to us to apply our mathematical sciences to them in any systematic fashion.

The nature of the Trinity, being three persons and yet one God, is beyond our comprehension. Our minds reflect His mind. As the light does not originate in that which reflects, we cannot comprehend God. Yet in as much as we bear the image of God, our reason can reflect the Logos. Note that where our mathematics is silent, as before the Trinity, God has given us other forms of His artwork in the creation to reflect the Trinity. For example, the covenantal body of Christ is one under the headship of her Lord.

Man's logic and mathematical mind are given to him to enable him to exercise dominion in the Lord over the earth. This is man's service of worship. Man was also created to speak the praises of God. This is man's word of worship. The mathematical eternal Logos

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2 Cornelius Van Til, *The Defense of the Faith*, (Philadelphia, Pennsylvania: The Presbyterian and Reformed Publishing Company, 1975), pp. 48-49.

is before His artistic display of mathematical thought in the creation which reflects His glory. Likewise, the image of the eternal mathematical Logos in our minds is necessary to comprehend mathematically the creation. It is expected then that the science of mathematics, which concerns the *a posteriori*; i.e., the artwork of God, presupposes the logos of mathematics, which is *a priori*; i.e., our thought which reflects the thought of God.

Bearing the image of the eternal mathematical Logos, man is able to know the creation which displays the glory of the Logos. Furthermore, man is able to know the creation mathematically to the extent that he bears that image and is thus taught by the Logos. Man is not created knowing the display of the mathematical, but able to know the mathematical. Man can therefore gain "new" knowledge. In actuality, he is discovering mathematics.

In the Socratic dialogue, *Meno*, Socrates argues that because a boy has discovered a mathematical truth without being told it, the soul must be eternal and learning merely recollection of knowledge already taught the soul. The position we have described above accounts for the discovery of new mathematical knowledge without recollection.

Indeed, man may recognize the reflection of the Logos in the creation in new ways. He is thus lead to discover new mathematics. But on the other hand, he may in his art discover a structure reflecting an aspect of the Logos only later to find it displayed in the creation. For example, the beginnings of what is now called fractal geometry were known mathematics in the early twentieth century. This of course was before the advent of high speed computation made possible the visual representation of the iteration of functions and the following recognition of such geometric patterns and shapes in the creation. When this recognition was made, what had been art was recognized as science, too. Thus, the science and art of mathematics are intertwined in ways and instances of which we are not fully cognizant. Our point here is that though we may all believe that the concept of the counting numbers is called to mind by counting ferns, we certainly do not claim that observing the geometry of a fern immediately calls to mind the concepts of affine transformations and matrix algebra. Experimentation with matrix theory and affine transformations was needed to represent the structure of the fern. The experimentation involved was experimentation with the artistic structures of mathematics itself. However, the mathematical structures of matrix algebra had already been developed for quite some time.

Indeed, the science of mathematics is possible only by the use of the art of mathematics. A question of science is which of our artistic representations best represents the created display of the Logos we are studying at the moment? Some humanistic mathematicians in the early twentieth century sought to build mathematics as a complete axiomatic system-one in which every true proposition would follow from a finite set of axioms without inconsistency. Professor Gödel in 1931 demonstrated by his incompleteness theorems that this goal is impossible. But the axiomatic art is not to be despised for its misuse by some. Proof within an incomplete system, the axioms of which are believed to reflect the Logos, is the “knowledge of faith” when applied in science to the creation. The above has been the position of this author for some time. However, I recently read the following lines in an article on mathematical proof and rigor in a professional mathematics journal. “The notion that absolute truth can be attained in mathematics goes back to Descartes and Leibniz in the 17th century. In the 19th century truth in mathematics was replaced by validity (relative truth) and, in the 20th century, by faith.”[3] We should ask today's mathematical scientist to explain to us in whom or what is his faith.

We thus come to understand that science is dependent upon art. The creation itself is the artwork of God to display His glory. The creation reflects the nature and thought of God. Not thinking the thoughts of God directly, we are to think thoughts which reflect His thoughts. It is our system of thought which reflects His thought by which we interpret the creation. And our mathematical thought is considered through our art of mathematics. Mathematical science is thus the application of mathematical art to the mathematical order of the creation for the purpose of exercising dominion in the Lord to the glory of God. By this we do not mean that mathematical science is the application of mathematical theory to empirical sciences. Applied mathematics, currently termed, is the modeling of empirical sciences in the theory of mathematical science. Instead, we mean that mathematical science, being the knowledge of the mathematical order of creation, is discovered by the artistic construction of mathematical structure and theory which reflects the mathematical order of the creation and, hence, ultimately the Logos. Man, bearing the image of the mathematical Logos, is able to construct his art which represents the art of God in creation. We represent this understanding by the schematic below.

Someone may object at this point that much if not most mathematics has historically been developed not deductively from a rigorous axiomatic point of view but intuitively, inductively, as well as deductively from a less rigorous conceptual framework. Such an objection is based on an accurate historical understanding, but it misses our point. We are not arguing that the mathematical learning process is one of only deduction from well-known axioms. However, the mathematician is working within a conceptual framework whether or not he has considered the foundation of that framework. Any mathematical observation and understanding presupposes a conceptual framework. And any mathematical argument presupposes a shared conceptual framework, or no argument would or could be made. To use even the concept of number is to cast one's work in the artwork of man. Otherwise, we would be thinking the very thoughts of God, which we cannot. Or, we must believe that number is somehow a property of reality in itself independent of God. Actually, all of our mathematics uses an artistic framework of mathematical concepts which we have developed and improved over time. Because the logos of our thought is in the image of God, our art can model the artistry of God. In this light consider the following comments of Professor Poythress.

> Our own mathematical systems (Euclidean or non-Euclidean) are somehow

3 Israel Kleiner, “Rigor and Proof in Mathematics: A Historical Perspective”, *Mathematics Magazine*, Vol. 64, No. 5 (1991), p. 307, footnote 38.

> not identical with His "system." We must say, I think, that Euclidean and non-Euclidean geometries are *both* exhibitions (revelations) of how God might rule the world; for they are both discoveries or constructions of the human mind in the *image* of God. Presumably God might have created a universe with either a Euclidean or non-Euclidean or some other geometry. Thus the variety of geometries, far from offering an obstacle to the Christian viewpoint, is simply an illustration of the freedom of God. (emphasis his)[4]

A proper and necessary use of the axiomatic system is to analyze the mathematical foundation of our mathematical, artistic framework. Such a system permits the construction of more precise concepts, definitions, and form, hopefully enabling us to avoid error. For example, the—concept and definition of a limit by Weierstrass enabled the avoidance of the confusion of convergence and uniform convergence which plagued Cauchy.[5] Secondly, more precise concepts and definitions make possible the expansion of the theory. We should add here not only the preciseness but also the "correctness" of concept and definition is needed. By "correctness" of a concept or definition we mean its reflection of the mathematical Logos. As an improperly focused glass diffuses the light, so does the unfocused concept make discovery a dim business. The correct concept with its corresponding definition however, like the magnifying glass that captures the light and intensifies all its rays on its object, enables man to see the mathematical clearly.

Thus mathematical learning involves both observation of the creation and experimentation and observation of the artwork of mathematics itself. As our artwork reflects the Logos, such experimentation can be fruitful. Thus purely mathematical concepts such as nonreal numbers, transcendental numbers such as **e** and , integrals, etc. can reflect the order of the creation. Thus the inductive characteristic of the mathematical process operates within the framework of the artists' concepts and constructs. Therefore, we understand how we derive our mathematical sustenance from God through the creation. The reflection of the Logos in the creation makes the observation of the creation fruitful for mathematical, artistic construction and inquiry.

Three concepts seem to be the core of the basis of mathematics as we know it. These are the concepts of number, set, and function. (Even though Zermelo-Fraenkel Set Theory provides a display of the concepts of number and function using the undefined terms of "set" and "element of", it is debatable whether the Theory has succeeded in reducing the concepts of function and number to that of set.)[6]

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4 Vern S. Poythress, "A Biblical View of Mathematics", pp. 185-186.

5 Israel Kleiner, "Rigor and Proof in Mathematics: A Historical Perspective", pp. 296-301.

6 Professor Poythress lists the sciences of kinematics, geometry, elementary algebra, and elementary set theory as the four sciences comprising mathematics. These sciences are concerned with movement, extension, number, and distinctness, respectively. It is this author's opinion that the transcendental aesthetic concepts of space and time are both mathematically constructed using the concepts of function, number, and set. For example, a Cartesian coordinate system is such a mathematical construct of the spatial. In such a coordinate system, the graph of a function with a numeric parameter representing time is the mathematical construct of the transcendental aesthetic concepts of space and time allowing for the mathematical modeling of the empirical sciences such as kinematics. See Vern S. Poythress, "A Biblical

In order to better understand how to develop these concepts in the student, we now turn in our discussion to Mathematics Education. Man is a mathematical person, but not every person is gifted as a mathematical artist. Therefore the early grades should emphasize experience. For example, teach counting by counting things! Have your young child begin by counting raisins. Teach subtraction by allowing them to eat the raisins. (Be sure to vary the food for a good diet and more fun.) After repeated experiences, ask your child questions to lead him or her to generalize their experience. Once they begin to see the patterns this way and generalize them, drill can periodically be used to insure mastery. There is no need to buy expensive math manipulatives to aid such experience. Raisins, marbles, and blocks work just fine.

Remember, young children are made in the image of God. Their minds are mathematical. Do not abuse this quality with boring manipulative tasks separated from investigative reasoning. For example, do not instruct your youngster, "we solve $x+3=6$ by subtracting 3 from both sides" and then give him 50 such problems to work. Instead say, "Suppose you had a certain amount of money, let's call it 'x' dollars. Suppose further that if you had three more dollars you would have 6 dollars. How many dollars must 'x' be?" The child will naturally solve this problem and others like it. Lead him by questions to list the steps he is using to solve the problem. Once the student has investigated the method of solution of such problems on his own, practice is in order to gain mastery.

In teaching, be sure to use analogies. For example, an equation may be thought of as a balanced scale. Hence, whatever you add or take from one side, you must also add or take from the other side.

For older children who have already learned to hate "math" due to the influence of educational formalism, special effort is needed. First, understand the enemy. Formalism is the view that mathematics is simply a game we play with rules we choose. Hence, mathematics is not taught in the early years from a concrete, practical, investigative approach, but from an approach that emphasizes abstract relationships or even "math facts" separated from experience. No one can live his life consistently holding to this view. For example, the exponential growth of compounded interest is not a game with no real consequences. Even as we cannot live our lives with such a view, we cannot successfully teach children from such a perspective. For those students who manage to learn the manipulative rules of the game, there is often a huge gap between what they view as mathematics (formal procedures) and experience. I believe that is one reason why so many college freshmen cannot do word problems. (Before we blame all of this problem on the humanistic schools however, let us remember that many parents demand that their children "succeed" in school despite the continued ignorance of the child. Our humanistic society has a pervasively "formal" view of education.)

For such children as these, begin by emphasizing the motivation for studying mathematics. We study mathematics to better exercise dominion in the Lord over the

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View of Mathematics", pp. 179-180.

earth and to glorify God. Next, begin teaching practical mathematics. (Now practical mathematics does not mean just addition and subtraction! The mathematics behind the design of a satellite which orbits the earth is very advanced beyond addition, yet exceedingly practical if you want telecommunications.) Start with whatever basic tools they lack, even if you have to go back to addition and multiplication. But teach the mathematics from a problem solving viewpoint before you lead them to generalize. For example, teach geometric series using the problem of finding the value of annuities. Do not first give them a formula for a geometric series to apply to an annuity. But lead them through the reasoning process with the concrete problem of finding the annuity's value and then guide them to generalize. This can be done. I have done it with college students who thought they could not do math. They get so involved in the process that they almost take over the class discussion. I like that.

Let me give another example. I recently spent an afternoon with my two older boys, ages 8 and 6, planting white pine trees eight feet apart along the back of our lot. During supper that evening, I mentioned to them that I had decided to plant a second row of pines to form equilateral triangles of 8 foot sides with the nearest pines in the first row. I then asked them to tell me how far apart the lines of the rows would be without having to go out in the dark to plant trees to find the answer. My six-year old at first blurted out, “Eight feet”, but just as quickly retracted his statement muttering that could not be right. My eight-year old held his head in his hands for a few moments and then said, “I will find out by drawing a triangle with each side one foot long and measure. Then I will multiply my result by eight.” I enjoyed watching my sons discover the Law of Similar Triangles this way. Later, we “discovered” the Pythagorean Theorem together. This theorem naturally called for the use of squares and square roots.

As children progress, they should begin to write mathematical arguments. Furthermore, all children should be taught basic logic, including an introduction to quantified predicate logic. The translation of English sentences into logic should be emphasized. Those especially gifted in mathematics can begin to study axiomatic systems, such as Euclidean geometry. I do not think that all children need to study geometry from a rigorous axiomatic viewpoint. I do believe that all should be trained in logic as discussed above. The age when a child should study these things varies with the child. Parents should be able to discover the understanding of their youngster.

From the early grades on, emphasize writing in mathematics! From the beginning, word problems should be answered in complete sentences. Require neat, ordered, reasoned steps in all work. Often ask the child to explain what he has done, and why he did it.

Math is often taught best in small groups. If possible, arrange for some group projects as well as individual assignments. The advantage of the home school or the small Christian school is that the mentor system can be used. The teacher is a mentor to ask questions to aid the student in his investigation of mathematics. Do not under estimate the importance of this method of teaching. Consider this comparison of math with, say, history. Can you lead a student purely by questions to discover on his own the reasons the Pilgrims left

England? Can he even discover in this way that there ever were Pilgrims? Yet mathematics can be discovered this way. For example, by questions you can lead a person to discover that an isosceles right triangle with legs of length one unit has an hypotenuse of length square root of two. To read how, see the Socratic dialogue, *Meno*.

We turn now to discuss the uses of the computer. The computer has its uses and its misuses. One particularly bad misuse is the substitution of “computer education” for mathematics education in the schools. Much of what is called “computer education” is merely training for computer use; that is, how to run the machine. Our choices of machines change. So do the procedures for using machines. Does anybody use punch cards anymore? What use would a computer class that taught you how to use a punch card machine be now? Besides, today's computers are incredibly easy to use. A child can learn to use many of their functions on his own. Such computer classes emphasize the obvious.

I believe based on personal experience as a teacher that another misconception is the belief that computer tutorial programs with their instant feedback are the ultimate teaching tool. They are useful, but not particularly better than traditional learning methods. My calculus class recently took an exam that was half conceptually based (definitions, use of theorems, proofs, analyzing the behavior of a function, etc.) and half techniques and applications (here is an integral-solve it, or here is an application-how would you prospective engineers handle it?) Three of my good students spent hours with some of the new tutorial software on the computer. The results were less than thrilling. On the other hand, another student worked problem after problem on her own and studied to understand how the theorems were used. She also spent time in my office reviewing the material. She scored far higher than any other student in the class for the first time this year. Note I wrote “she”. Do not believe it if someone tells you that girls can't do math.

However, even though the computer is not a teacher in itself, it is an excellent tool for mathematical experimentation! Especially graphic experiments are helpful. Have the computer graph y=sin2(x). Then have it graph y=cos2(x). Then have it graph y=sin2(x) +cos2(x). Students will see and believe that sin2(x) + cos2(x) = 1. Then lead them in a proof of it. Or graph twenty general linear equations Ax + By = C where A, B, and C are in arithmetic progression. See what you get. Explain why it happens.

Another great use of the computer is to eliminate boring manipulations. This allows for better applications, especially in such fields as Linear Algebra.

However, expensive lab equipment is not necessary, especially in the younger grades. You don't think that an electron microscope is necessary to teach Johnny biology do you? A much cheaper microscope will do to see a cell. Likewise, a graphics calculator (about $90, tops) should suffice in math. And if you cannot afford that, don't worry about it. Isaac Newton discovered the calculus long before computers came on the scene.

If your child evidences extraordinary talent in an area, use a tutor to help develop that talent when necessary. One of the reasons we have society is the division of labor. Use the expertise of others wisely.

For further reading on the subject of the philosophy of mathematics, I suggest the excellent article by Dr. Vern Poythress, “A Biblical View of Mathematics”, in *Foundations of Christian Scholarship*, edited by Gary North, Ross House Books, 1979. This author is indebted to Professor Poythress's teaching on this subject through his paper named above. For a historical view of the various philosophies of mathematics, see “Rigor and Proof in Mathematics: A Historical Perspective”, by Israel Kleiner, in the December 1991 issue of *Mathematics Magazine*, published by The Mathematical Association Of America, 1529 Eighteenth Street, N.W., Washington D.C. 20036-1385.

Some questions which I put to the mathematically and philosophically inclined reader for further discussion are the following. Regarding pathological examples, such as an everywhere continuous but nowhere differentiable function, are these examples illustrations of the inadequacies of our artistic, conceptual constructs? Or, are they to us what the discovery of the irrationality of the square root of 2 was to the ancient Greeks? That is, are some of these pathological examples not pathological at all, but glimpses of the constructs of a branch of fruitful mathematics yet to be discovered? Will we have to wait over 1500 years to find out as was the case with irrationals? What are the criteria by which we judge the degree of conformity of our artistic constructs to the Logos? Is present day usefulness alone the criterion? Or the possibility of usefulness? How do we judge, since with a paradigm change in science what was considered useless can become the construct of choice? What are the aesthetic criteria of mathematics? Work on the development of the biblical theory of mathematical proof, or justification, of a result would be worthwhile. What is proof, or justification, of a result is debated today with the question of what is a computer “proof” added to the discussion.

Reformed mathematicians have begun to recognize that answers for the philosophy of mathematics must be found ultimately in the nature and will of God. Descriptions of the biblical approach have been given. But, as far as I know, we have not approached any of these questions in a systematic way. There is work to be done.